COHOMOLOGY FOR FROBENIUS KERNELS OF SL_2

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ABSTRACT. Let $(SL_2)_r$ be the r-th Frobenius kernels of the group scheme SL_2 defined over an algebraically field of characteristic $p > 2$. In this paper we give for $r \geq 1$ a complete description of the cohomology groups for $(SL_2)_r$. We also prove that the reduced cohomology ring $\mathrm{H}^\bullet((SL_2)_r, k)_{\mathrm{red}}$ is Cohen-Macaulay. Geometrically, we show for each $r \geq 1$ that the maximal ideal spectrum of the cohomology ring for $(SL_2)_r$ is homeomorphic to the fiber product $G \times_B \mathfrak{u}^r$. Finally, we adapt our calculations to obtain analogous results for the cohomology of higher Frobenius-Luzstig kernels of quantized enveloping algebras of type SL_2.

1. Introduction

1.1. In recent years, the cohomology and representation theory of Frobenius kernels has received considerable interest. It is well-known that the category of restricted representations for the Lie algebra of an algebraic group in characteristic $p > 0$ is equivalent to that of representations over the first Frobenius kernels of the algebraic group. This connection has inspired many investigations of the cohomology for the Frobenius kernels of algebraic groups. For higher Frobenius kernels, the cohomology is of interest because it provides information about the original group scheme. For instance, let G be a simple, simply-connected algebraic group defined over an algebraically closed field k of characteristic $p > 0$. Denote by G_r the r-th Frobenius kernel of G. Then for each $i \geq 0$, we can identify $\mathrm{H}^i(G, M)$ with the inverse limit $\varprojlim_r \mathrm{H}^i(G_r, M)$ under some mild assumptions on the given finite dimensional G-module M [Jan, I.9]. In 2006, Bendel, Nakano, and Pillen computed the first and second degree cohomology of the r-th Frobenius kernels of G [BNP1],[BNP2]. However, the state of affairs for higher degree cohomology remains an open problem in general.

Via geometry, more is known about the cohomology of Frobenius kernels. It was first noticed for semisimple algebraic groups by Friedlander and Parshall that there is an isomorphism between the cohomology ring for G_1 and the coordinate ring of the nilpotent cone for the Lie algebra $\mathfrak{g}$ =Lie(G) [FP]. A generalization was obtained for the higher Frobenius kernels by Suslin, Friedlander, and Bendel [SFB1], [SFB2]. They constructed a homeomorphism between the spectrum of the G_r-cohomology ring and the variety of r-tuples of commuting nilpotent elements in $\mathfrak{g}$. Moreover, in the case $G = SL_2$, Suslin, Friedlander, and Bendel explicitly computed the support varieties for the induced module and simple module of every dominant weight [SFB2, Proposition 6.10].

In the view of cohomology for finite groups, Benson and Carlson studied the algebraic structure of cohomology rings [Car]. One of their major concerns is whether the cohomology ring of a finite group is Cohen-Macaulay. It is known that if E is an elementary abelian p-group, then the cohomology ring for E is Cohen-Macaulay [Ben1, 5.18]. In this paper we are concerned with the same question applied to the cohomology of Frobenius kernels. Since the cohomology ring for the first Frobenius kernel of G is isomorphic to the coordinate ring of the nullcone, it is Cohen-Macaulay. Our proof in Section 7 establishes the Cohen-Macaulayness of the cohomology rings for the higher Frobenius kernels of SL_2.

1.2. **Main results.** The paper is organized as follows. We establish basic notation in Section 2. Then in Section 3, we derive an explicit spectral sequence to compute the cohomology of B_r from

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the spectral sequence stated in [Jan, I.9.14] and recall a strategy to calculate the cohomology of G_r.

Section 4 contains cohomology calculations for the subgroup schemes U_r, B_r, and G_r of SL_2 where B is a fixed Borel subgroup of G and U is its unipotent radical subgroup. The computation for cohomology of G_r provides a new proof of a result of van de Kallen concerning good filtrations on the cohomology groups $\mathrm{H}^n((SL_2)_r, \mathrm{H}^0(\lambda))$ with $n \geq 0$ and arbitrary dominant weight λ [vdK]. One will notice that the results in this section depend highly on the multiplicity of certain dominant weights. Hence, in the next section we introduce an algorithm that calculates the character multiplicity of a weight μ in $\mathrm{H}^\bullet(B_r, \lambda)$, and hence the multiplicity of $\mathrm{H}^0(\mu)$ in $\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))$. In Appendix, we provide computer calculations showing that our algorithm is faster than the one encoded by using Ehrhart polynomial. We compute the rings $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ and $\mathrm{H}^\bullet(G_r, k)_{\mathrm{red}}$ in Section 6 in order to investigate their geometric structures. In the process, we develop several techniques to study reduced commutative rings. This computation shows that there is a homeomorphism between the spectra of the reduced G_r-cohomology ring and the ring of global sections on $G \times^B \mathfrak{u}^r$ (cf. Proposition 6.2.4). This also plays a key role in showing that the reduced B_r- and G_r-cohomology rings are Cohen-Macaulay (Section 7). This new result inspires us to state a conjecture on the Cohen-Macaulayness of the cohomology ring for the Frobenius kernels of an algebraic group.

The last section is devoted to studying analogous results for the higher Frobenius-Lusztig kernels of quantum groups. These objects, which are analogs for the hyperalgebras of the higher Frobenus kernels of an algebraic group, were first defined by Drupieski in [Dru] in the context of quantized enveloping algebras defined over fields of positive characteristic. He verified various properties of the Frobenius-Lusztig kernels which are similar to the results obtained in the case of characteristic 0.

2. Notation

2.1. **Representation theory.** Let k be an algebraically closed field of characteristic $p > 2$. Let G be a simple, simply-connected algebraic group over k, defined and split over the prime field $\mathbb{F}_p$. Denote by h the Coxeter number of G. Fix a maximal torus T of G, and denote by Φ the root system of T in G. Fix a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots in Φ, and let Φ^+ be the corresponding set of positive roots. Let $B \subset G$ be the Borel subgroup of G containing T and corresponding to Φ^+, the set of positive roots, and let $U \subset B$ be the unipotent radical of B.

Let W be the Weyl group of Φ; it is generated by the set of simple reflections $\{s_\alpha : \alpha \in \Pi\}$. Write $\ell : W \to \mathbb{N}$ for the standard length function on W, and let $w_0 \in W$ be the longest element. Let $(\cdot, \cdot)$ be the standard W-invariant inner product on the Euclidean space $\mathbb{E} := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Given $\alpha \in \Phi$, let $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Set α_0 to be the highest short root of Φ, and ρ to be one-half the sum of all positive roots in Φ. Then the Coxeter number of Φ is $h = (\rho, \alpha_0^\vee) + 1$. Suppose $\lambda = \sum_{\alpha \in \Pi} m_\alpha \alpha$ a weight in X, then the height of λ is defined as $\mathrm{ht}(\lambda) = \sum_{\alpha \in \Pi} m_\alpha$.

Let X be the weight lattice of Φ, defined by the $\mathbb{Z}$-span of the fundamental weights $\{\omega_1, \ldots, \omega_n\}$, and let $X^+ \subset X$ be the set of dominant weights. Simple G-modules are indexed by $\lambda \in X^+$, and denoted by $L(\lambda)$. The simple module $L(\lambda)$ can be identified with the socle of the induced module $H^0(\lambda) = \mathrm{ind}_{B^-}^G \lambda$. Set $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G, $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{u} = \mathrm{Lie}(U)$. Denote by $S^\bullet(\mathfrak{u}^*)$ and $\Lambda^\bullet(\mathfrak{u}^*)$ respectively the symmetric algebra and exterior algebra over $\mathfrak{u}^*$. Throughout this paper, the symbol $\otimes$ means the tensor product over the field k, unless otherwise stated. Suppose H is an algebraic group over k and M is a (rational) module of H. Denote by M^H the submodule consisting of all the fixed points of M under the H-action.

For each r, the r-th Frobenius kernel of a closed subgroup H of G is defined as the intersection of H with the kernel of the morphism $F_r : G \to G$ induced from the ring endomorphism $f \mapsto f^{p^r}$ on $k[G]$. Given a rational H-module M, write $M^{(r)}$ for the module obtained by twisting the structure map for M by F_r. Now suppose $R \subseteq S$ are finitely generated commutative algebras over k. Let

$F_r(S) = \{s^{p^r} \mid s \in S\}$ for every $r \geq 1$. If we have the following inclusions

$$F_r(S) \subseteq R \subseteq S$$

then the inclusion $R \hookrightarrow S$ induces the homeomorphism from Spec S onto Spec R, and we call this map an F-isomorphism.

2.2. **Geometry.** Let R be a commutative Noetherian ring with identity. We use R_{red} to denote the reduced ring $R/\operatorname{Nilrad} R$ where $\operatorname{Nilrad} R$ is the radical ideal of 0 in R, which consists of all nilpotent elements of R. Let Spec R be the spectrum of all prime ideals of R. This set is a topological space under the Zariski topology. Let X be a variety. We denote by $k[X]$ the algebra of regular functions defined on X. Note that when X is an affine variety, $k[X]$ coincides with the coordinate algebra of X.

Denote by $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. There is an adjoint action of G on $\mathfrak{g}$ which stablizes $\mathcal{N}$. We call it the dot action and use "$\cdot$" for the notation. Then for every element $v \in \mathcal{N}$ we let $\mathcal{O}_v$ denote the G-orbit of v (i.e., $\mathcal{O}_v = G \cdot v$). For example, the orbits $\mathcal{O}_{\text{reg}} = G \cdot v_{\text{reg}}, \mathcal{O}_{\text{subreg}} = G \cdot v_{\text{subreg}}$ and $\mathcal{O}_{\text{min}} = G \cdot v_{\text{min}}$ are repsectively corresponding to the sets of regular, subregular and minimal elements in $\mathcal{N}$. For H a subgroup of G, suppose X, Y are H-varieties. Then the morphism $f : X \to Y$ is called H-equivariant if it is compatible with H-action.

Given a G-variety V, it can be seen that B acts freely on $G \times V$ by setting $b \cdot (g, v) = (gb^{-1}, bv)$ for all $b \in B, g \in G$ and $v \in V$. The notation $G \times^B V$ stands for the fiber bundle associated to the projection $\pi : G \times^B V \to G/B$ with fiber V. Topologically, $G \times^B V$ is a quotient space of $G \times V$ in which the equivalence relation is given as

$$(g, v) \sim (g', v') \Leftrightarrow (g', v') = b \cdot (g, v) \text{ for some } b \in B.$$

In other words, each equivalence class of $G \times^B V$ represents a B-orbit in $G \times V$. The map $m : G \times^B V \to G \cdot V$ defined by mapping $[g, v]$ to $g \cdot v$ for all $g \in G, v \in V$ is called the moment morphism. It is obviously surjective. Although $G \times^B V$ is not affine, we still denote by $k[G \times^B V]$ the ring of global sections on this variety. It is sometimes useful to make the following identification: $k[G \times^B V] \cong k[G \times V]^B$. Note also that Spec $k[G \times^B V] = G \times_B V$.

3. Main tools

In this section, we introduce some methods to compute B_r and G_r-cohomology which will be applied later to obtain results on $(SL_2)_r$-cohomology. For simplicity, we write S^i and Λ^j instead of $S^i(\mathfrak{u}^*)$ and $\Lambda^j(\mathfrak{u}^*)$. We start with the spectral sequence in [Jan, Proposition I.9.14]. Replacing $\mathfrak{g}^*$ by $\mathfrak{u}^*$, we immediately get a new spectral sequence to compute cohomology of U_r with coefficients in a B-module M. The resulting spectral sequence can be written as follows:

$$E_1^{i,j} = \bigoplus M \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)} \Rightarrow \mathrm{H}^{i+j}(U_r, M) \tag{1}$$

where the direct sum is taken over all a_i's and b_j's satisfying

$$\begin{cases} i + j &= 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r \\ i &= \sum_{n=1}^{r} (a_n p^n + b_n p^{n-1}). \end{cases} \tag{2}$$

This spectral sequence will play an important role in our calculations for cohomology in later sections.

3.1. **Spectral sequence for B_r-cohomology.** In order to set up an inductive proof and some intuition for the proofs, we first look at a simple case when $r = 2$. As T_2 is diagonalisable, the fixed point functor $(-)^{T_2}$ is exact. Hence,

$$\mathrm{H}^{i+j}(U_2, M)^{T_2} \cong \mathrm{H}^{i+j}(B_2, M).$$

So applying the fixed point functor on both sides of the spectral sequence (1), we have

$$\bigoplus \left(M\otimes S^{a_1(1)}\otimes S^{a_2(2)}\otimes \Lambda^{b_1}\otimes \Lambda^{b_2(1)}\right)^{T_2} \Rightarrow \mathrm{H}^{i+j}(U_2,M)^{T_2}$$

$$\bigoplus \left(S^{a_1(1)}\otimes S^{a_2(2)}\otimes (M\otimes \Lambda^{b_1})^{T_1}\otimes \Lambda^{b_2(1)}\right)^{T_2/T_1} \Rightarrow \mathrm{H}^{i+j}(B_2,M)$$

$$\bigoplus \left(S^{a_1}\otimes S^{a_2(1)}\otimes (M\otimes \Lambda^{b_1})^{T_1(-1)}\otimes \Lambda^{b_2}\right)^{T_1^{(1)}} \Rightarrow \mathrm{H}^{i+j}(B_2,M)$$

$$\bigoplus S^{a_2(2)}\otimes \left(S^{a_1}\otimes \Lambda^{b_2}\otimes (M\otimes \Lambda^{b_1})^{T_1(-1)}\right)^{T_1^{(1)}} \Rightarrow \mathrm{H}^{i+j}(B_2,M)$$

where each direct sum is taken over all a_i, b_i's satisfying the condition (2). This computation can be generalized for arbitrary r as follows.

Theorem 3.1.1. *There exists, for each B-module M, a spectral sequence converging to $\mathrm{H}^\bullet(B_r,M)$ as a $B^{(r)}$-module with the first page:*

$$E_1^{i,j} = \bigoplus S^{a_r(r)}\otimes \left[\left[(M\otimes\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes \Lambda^{b_2}\right]^{T_1(-1)}\otimes ..\otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}} \tag{3}$$

where the direct sum is taken over all a_i, b_j satisfying condition (2). Alternatively, we can write

$$\bigoplus_{n=2(a_1+..+a_r)+b_1+..+b_r} S^{a_r(r)}\otimes \left[\left[(M\otimes\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes \Lambda^{b_2}\right]^{T_1(-1)}\otimes ..\otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}} \Rightarrow \mathrm{H}^n(B_r,M)$$

Proof. We first consider the spectral sequence for U_1-cohomology as follows:

$$\bigoplus M\otimes S^{a_1(1)}\otimes \Lambda^{b_1} \Rightarrow \mathrm{H}^n(U_1,M)$$

Taking T_1-invariant functor on both sides, we have

$$\bigoplus (M\otimes S^{a_1(1)}\otimes \Lambda^{b_1})^{T_1} \Rightarrow (\mathrm{H}^n(U_1,M))^{T_1} \cong \mathrm{H}^n(B_1,M)$$

$$\bigoplus S^{a_1(1)}\otimes (M\otimes \Lambda^{b_1})^{T_1} \Rightarrow \mathrm{H}^n(B_1,M).$$

This verifies the theorem for $r=1$. Suppose it is true for r. Apply the invariant functor $(-)^{T_{r+1}}$ on the U_{r+1}-spectral sequence, we obtain

$$\bigoplus \left(M\otimes S^{a_1(1)}\otimes ..\otimes S^{a_{r+1}(r+1)}\otimes \Lambda^{b_1}\otimes ..\otimes \Lambda^{b_{r+1}(r)}\right)^{T_{r+1}} \Rightarrow \mathrm{H}^{i+j}(U_{r+1},M)^{T_{r+1}} \cong \mathrm{H}^{i+j}(B_{r+1},M)$$

with a_i, b_j satisfying

$$\begin{cases} i+j &= 2(a_1+\cdots+a_{r+1})+b_1+\cdots+b_{r+1} \\ i &= \sum_{n=1}^{r+1}(a_np^n+b_np^{n-1}). \end{cases}$$

In order to complete our induction proof on r, we show that the E_1-page of this above spectral sequence can be rewritten in the form of (3). First note that $T_j/T_i \cong T_{j-i}^{(i)}$ for $0\le i\le j$ where $T_0 = T$. So if M is a T-module, it can be identified with a T/T_i-module for each i via the aforementioned isomorphism. In particular, we have

$$M^{T_j/T_i} \cong M^{T_{j-i}^{(i)}} \cong \left(M^{(-i)}\right)^{T_{j-i}}$$

for each $0 \le i \le j$. This observation gives us the following implications on each direct summand of LHS:

$$\left[\left(M \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \cdots \otimes \Lambda^{b_r(r-1)}\right)^{T_r} \otimes S^{a_{r+1}(r+1)} \otimes \Lambda^{b_{r+1}(r)}\right]^{T_{r+1}/T_r}$$
$$\cong \left[\left(M \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \cdots \otimes \Lambda^{b_r(r-1)}\right)^{T_r(-r)} \otimes S^{a_{r+1}(1)} \otimes \Lambda^{b_{r+1}}\right]^{T_1^{(r)}}$$
$$\cong \left[\left(S^{a_r(r)} \otimes \left[\left[(M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}}\right)^{(-r)} \otimes S^{a_{r+1}(1)} \otimes \Lambda^{b_{r+1}}\right]^{T_1^{(r)}}$$
$$\cong \left[S^{a_r} \otimes \left[\left[(M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes \Lambda^{b_r}\right]^{T_1^{(-1)}} \otimes S^{a_{r+1}(1)} \otimes \Lambda^{b_{r+1}}\right]^{T_1^{(r)}}$$
$$\cong \left[\left[\left[(M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes \Lambda^{b_r}\right]^{T_1(-1)} \otimes S^{a_r} \otimes \Lambda^{b_{r+1}}\right]^{T_1^{(r)}} \otimes S^{a_{r+1}(r+1)}$$

where the second isomorphism is by inductive hypothesis. This completes our proof. □

3.2. **Spectral sequence for G_r-cohomology.** Recall form [Jan, II.12.2] that if $R^i \operatorname{ind}_B^G M = 0$ for all $m > 0$, then G_r-cohomology can be computed from the following spectral sequence

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, M)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \operatorname{ind}_B^G M)^{(-r)}.$$

In particular, for any dominant weight $\lambda \in X^+$, we always have

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, \lambda)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \mathrm{H}^0(\lambda))^{(-r)}.$$

So our strategy is to compute $\mathrm{H}^m(B_r, \lambda)$ first by Theorem 3.1.1, then use this spectral sequence to get G_r-cohomology of $\mathrm{H}^0(\lambda)$.

4. Cohomology

We assume from now on that $G = SL_2$ and p is an arbitrary odd prime. Let α be the only simple root in the root system Φ of G. Denote ω the fundamental weight in the weight lattice X [Hum, 13.2]. Then we have $\omega = \frac{\alpha}{2}$. Note also that $\mathfrak{u}$ is a one-dimensional vector space so we have the following T-module identifications on each degree of the exterior algebra

$$\Lambda^i = \Lambda^i(\mathfrak{u}^*) = \begin{cases} k & \text{if } i = 0, \\ \alpha & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Our overall goal is computing the cohomology $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))$ for every dominant weight $\lambda \in X^+$. Following the strategy in Subsection 3.2, we start with U_r-cohomology.

4.1. Cohomology of U_r.

Proposition 4.1.1. *Let λ be a dominant weight. For each $r \ge 1$, there is a B-isomorphism*

$$\mathrm{H}^n(U_r, \lambda) \cong \bigoplus_{n=2(a_1+\cdots+a_r)+b_1+\cdots+b_r} \lambda \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)}.$$

Proof. Recall from Section 3 that U_r-cohomology can be computed by the following spectral sequence

$$E_1^{i,j} = \bigoplus \lambda \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)} \Rightarrow \mathrm{H}^{i+j}(U_r, \lambda)$$

where the direct sum is taken over all tuples $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying $i + j = 2(a_1 + \ldots + a_r) + b_1 + \ldots + b_r$ and $i = \sum_{n=1}^{r}(a_n p^n + b_n p^{n-1})$. Observe that as a B-module, $S^m = m\alpha$ and $\Lambda^0 = k, \Lambda^1 = \alpha$, and $\Lambda^m = 0$ for $m > 1$. Consider for each $n > 0$, we have $d_n^{i,j} : E_n^{i,j} \to E_n^{i+n,j-n+1}$ where the B-module $E_n^{i,j}$ has weight

$$\begin{aligned}\lambda + pa_1\alpha + \cdots + p^r a_r\alpha + b_1\alpha + \cdots + p^{r-1}b_r\alpha &= \lambda + (pa_1 + \cdots + p^r a_r + b_1 + \cdots + p^{r-1}b_r)\alpha \\ &= \lambda + i\alpha.\end{aligned}$$

Likewise, $E_n^{i+n,j-n+1}$ is of weight $\lambda + (i+n)\alpha$. As all the differentials respect to T-action, we must have $\lambda + i\alpha = \lambda + (i+n)\alpha$ if the map is nonzero. This implies $n = 0$, and so $d_n^{i,j} = 0$ for all i, j and $n > 0$. Hence the spectral sequence collapses at the first page. The result therefore follows. □

When $\lambda = 0$, the isomorphism is also compatible with the ring structure. This computation was completely done by Andersen-Jantzen in [AJ, 2.4]. We recall their result as follows.

Corollary 4.1.2. *For each $r \geq 1$, there is an isomorphism of B-algebra*

$$\mathrm{H}^\bullet(U_r, k) \cong S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^\bullet \otimes \cdots \otimes \Lambda^{\bullet(r-1)}.$$

Consequently,

$$\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}} \cong S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)}.$$

Remark 4.1.3. Another way to obtain the isomorphism is to exploit the fact that U_r is an abelian group which is the product $U_1 \times U_1^{(1)} \times \cdots \times U_1^{(r-1)}$. The theory of group cohomology gives us a complete description of $\mathrm{H}^\bullet(U_r, k)$ as a ring. Our proof actually has more flavors than that. It guarantees the isomorphism is compatible with the B-module structure which will become handy in computing G_r-cohomology later.

4.2. **Cohomology of B_r.** For later convenience, we identify $S^{\bullet(i)}$ with the polynomial ring $k[x]^{(i)} = k[x_i]$ where x_i has weight $p^i\alpha$ and degree 2. We also denote for each $1 \leq i \leq r$ by y_{i-1} the generator of the exterior algebra $\Lambda^{(i)}$. In particular, we have

$$S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^\bullet \otimes \cdots \otimes \Lambda^{\bullet(r-1)} = k[x_1, \ldots, x_r] \otimes \Lambda(y_0, \ldots, y_{r-1}).$$

Now we make use of this notation to describe the B_r-cohomology.

Proposition 4.2.1. *Suppose λ is a dominant weight in X^+, i.e., $\lambda = m\omega$ for some non-negative integer m. For each $r \geq 1$, there is a B-isomorphism*

$$\mathrm{H}^n(B_r, \lambda)^{(-r)} \cong \bigoplus \left\langle x_1^{a_1} y_0^{b_1} x_2^{a_2} y_1^{b_2} \cdots x_r^{a_r} y_{r-1}^{b_r} \right\rangle$$

where the direct sum is taken over all a_i, b_j satisfying the following conditions

$$\begin{cases} a_i \in \mathbb{N} \text{ and } b_i \in \{0,1\} \text{ for all } 1 \leq i \leq r \\ n = 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r \\ \frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + a_r p^r \in p^r X^+. \end{cases} \tag{4}$$

Proof. It is observed that collapsing of the spectral sequence (1) implies the collapse of the one in Theorem 3.1.1. So Proposition 4.1.1 implies that

$$\mathrm{H}^n(B_r, \lambda) \cong \bigoplus S^{a_r(r)} \otimes \left[\left[(\lambda \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes S^{a_{r-1}} \otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}}$$

where the direct sum is taken over all tuples $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying $n = 2(a_1 + \ldots + a_r) + b_1 + \ldots + b_r$. Using the identifications earlier, we can explicitly write out the cohomology for B_r as a decomposition of weight spaces

$$\mathrm{H}^n(B_r, \lambda)^{(-r)} \cong \bigoplus \left\langle x_1^{a_1} y_0^{b_1} x_2^{a_2} y_1^{b_2} \cdots x_r^{a_r} y_{r-1}^{b_r} \right\rangle$$

where each monomial weights

$$\left[a_r\alpha + \cfrac{\cfrac{\cfrac{\frac{\lambda+b_1\alpha}{p}+(a_1+b_2)\alpha}{p}+(a_2+b_3)\alpha}{p} + \cdots + (a_{r-1}+b_r)\alpha}{p} \right] \in X^+.$$

This is equivalent to $\frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + a_r p^r \in p^r X^+$. Now as U is an abelian group in this case, it trivially acts on both sides of the above isomorphism. Hence, this isomorphism is compatible with B-action via the identification $B/U \cong T$. $\square$

Remark 4.2.2. For each tuple $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying (4), there is $\gamma = n\omega$ for some non-negative integer n such that

$$\frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + (a_{r-1} + b_r)p^{r-1} + a_r p^r = \frac{n}{2} p^r. \tag{5}$$

Let $N_r(p, m, n)$ denote the number of solutions $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ for the equation. As a consequence, we establish our goal of this subsection.

Theorem 4.2.3. *Suppose $\lambda = m\omega \in X^+$. Then there is a B-module isomorphism*

$$\mathrm{H}^\bullet(B_r, \lambda)^{(-r)} \cong \bigoplus_{n=0}^{\infty} (n\omega)^{N_r(p,m,n)}$$

and so

$$\mathrm{ch}\, \mathrm{H}^\bullet(B_r, \lambda)^{(-r)} = \sum_{n \in \mathbb{N}} N_r(p, m, n) e(n\omega).$$

4.3. **Cohomology of G_r.** We are now ready to compute G_r-cohomology.

Theorem 4.3.1. *Suppose $\lambda = m\omega \in X^+$. Then there are G-module isomorphisms*

$$\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^\bullet(B_r, \lambda)^{(-r)}) \cong \bigoplus_{n=0}^{\infty} \mathrm{ind}_B^G\, (n\omega)^{N_r(p,m,n)}.$$

where each $N_r(p, m, n)$ is defined in Subsection 5. In particular, if $\lambda = 0$, then the first isomorphism is compatible with the cup-products on both sides, i.e., it is an isomorphism of graded G-algebras.

Proof. Recall the spectral sequence in Section 3.2, we have

$$E_2^{m,n} = R^m \mathrm{ind}_B^G(\mathrm{H}^n(B_r, \lambda)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \mathrm{H}^0(\lambda))^{(-r)}.$$

Note that all weights of $\mathrm{H}^\bullet(B_r, \lambda)$ are also weights of $\mathrm{H}^\bullet(U_r, \lambda)$ which are in turn weights of $\lambda \otimes S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^\bullet \otimes \Lambda^{\bullet(1)} \otimes \cdots \otimes \Lambda^{\bullet(r-1)}$ by the spectral sequence (1). So all weights of $\mathrm{H}^n(B_r, \lambda)$ are dominant for each $n \geq 0$. It follows from Kempf's vanishing that $R^m \mathrm{ind}_B^G(\mathrm{H}^n(B_r, \lambda)) = 0$ for each $n \geq 0$ and $m > 0$. Hence the spectral sequence collapses at the first page and we obtain

$$\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G \left(\mathrm{H}^n(B_r, \lambda)^{(-r)} \right).$$

This implies the first isomorphism in the theorem, that is,

$$\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^\bullet(B_r, \lambda)^{(-r)}).$$

Moreover, by [AJ, Remark 3.2], this isomorphism respects the graded $\mathrm{H}^n(G_r, k)$-module structure on both sides. Hence, in the case $\lambda = 0$ it is a graded G-algebra isomorphism. The other isomorphism follows by Theorem 4.2.3. □

Remark 4.3.2. This theorem gives us an explicit good filtration of $\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))^{(-r)}$; thus, showing the same property for each $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))^{(-r)}$ with $n \geq 0, r \geq 1$. This result is similar to the one in [AJ, 4.5(1)] and [vdK, Corollary 2.2]. In fact, for arbitrary simple algebraic group G, it is conjectured that $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))$ has a good filtration [Jan, 12.15].

Following Theorem 4.2.3, we immediately obtain the analogous result for G_r-cohomology.

Corollary 4.3.3. *Suppose $\lambda = m\omega \in X^+$. Then we have*

$$\operatorname{ch} \mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))^{(-r)} = \sum_{n \in \mathbb{N}} N_r(p, m, n) \operatorname{ch} \mathrm{H}^0(n\omega).$$

5. An algorithm to compute $N_r(p, m, n)$

5.1. Results in the preceding section indicate that the number $N_r(p, m, n)$ plays an important role in the cohomology of B_r and G_r. It is closely related to the problem of counting integral lattice points in a polytope. In particular, let P_r be a polytope in $\mathbb{R}^r$ determined by the following equations:

$$\begin{cases} \frac{m}{2} + y_1 + (x_1 + y_2)p + \cdots + (x_{r-1} + y_r)p^{r-1} + x_r p^r = \frac{n}{2}p^r, \\ 0 \leq x_i \leq \frac{n}{2}p^r \text{ for each } 1 \leq i \leq r, \\ 0 \leq y_i \leq 1 \text{ for each } 1 \leq i \leq r. \end{cases}$$

Then we have $N_r(p, m, n) = |P_r \cap \mathbb{Z}^r|$ for each $r \geq 1$. So one can use Barvinok's algorithm to compute the right-hand side. However, this algorithm is getting slow when r is big due to many complicated subalgorithms and computations involving Complex Analysis. Barvinok actually proved that the algorithm terminates after a polynomial time for a fixed dimension depending on the data for vertices of the polytope (see [Bar] for further details).

5.2. In this subsection, we sketch an alternative program to calculate the number $N_r(p, m, n)$ with given non-negative integers m, n; $r \geq 2$ and prime $p > 2$. We start with an algorithm to compute the number of solutions $(c_1, ..., c_r)$ satisfying the equation

$$np^r = (c_1 + d_1)p + ... + (c_r + d_r)p^r \tag{6}$$

with given nonnegative integer d_i for each i. Define recursively a family of functions $N_i : p^i\mathbb{N} \to \mathbb{N}$ as follow:

- $N_i(0) = 1$ for each $1 \leq i \leq r$
- For each $n \in \mathbb{N}$, $N_1(np) = \begin{cases} 0 & \text{if} \quad d_1 > n \\ 1 & \text{otherwise} \end{cases}$
- For each $n \in \mathbb{N}$, $N_2(np^2) = \begin{cases} 0 & \text{if} \quad d_2 > n \\ \Sigma_{j=0}^{n-d_2} N_1(jp^2) & \text{otherwise} \end{cases}$
- For each $n \in \mathbb{N}$, $N_{i+1}(np^{i+1}) = \begin{cases} 0 & \text{if} \quad d_{i+1} > n \\ \Sigma_{j=0}^{n-d_{i+1}} N_i(jp^{i+1}) & \text{otherwise} \end{cases}$

Theorem 5.2.1. *For each $n \in \mathbb{N}$, the number of solutions of equation* (6) *is $N_r(np^r)$.*

Proof. We prove by induction on r. It is trivial for $r = 1$. Suppose it is true for $r - 1$. From equation (6), we have $n - d_r + 1$ choices for $c_r \in \{d_r, d_r + 1, .., n\}$. For each $c_r = i$, the number of solutions is $N_{r-1}(np^r - ip^r) = N_{r-1}[p(n-i)p^{r-1}]$. Summing all the terms and using the recursive formula, we get the total number of solutions $N_r(np^r)$. □

Now the algorithm consists of following steps:

- If m is even and n is odd, or if m is odd and n is even then the function returns 0.
- If both m, n are even, then let $m' = \frac{m}{2}$ and $n' = \frac{n}{2}$. The equation 5 hence becomes

$$m' + b_1 + (a_1 + b_2)p + ... + (a_{r-1} + b_r)p^{r-1} + a_r p^r = n' p^r. \tag{7}$$

  (a) Write m' into base p-expansion. Suppose $m' = d_0 + d_1 p + ... + d_h p^h$ for some $h \in \mathbb{N}$. Note that if $b_1 + m' > n' p^r$ then, of course, $N_r(p, m, n) = 0$.
  (b) For each r-tuple $(b_1, ..., b_r) \in \{0, 1\}^r$, we use previous theorem to compute number $N_r(m', n', b_1, ..., b_r)$ of solutions for

$$d_0 + b_1 + (a_1 + d_1 + b_2)p + ... + (a_{r-1} + d_{r-1} + b_r)p^{r-1} + (a_r + d_r)p^r = (n' - d_{r+1}p - ... - d_h p^{h-r})p^r.$$

  (c) We have $N_r(p, m, n) = \Sigma_{b_1, .., b_r} N_r(m', n', b_1, ..., b_r)$.
- If both m and n are odd, then repeat Step 2 with $m' = \frac{m+1}{2}$ and $n' = \frac{n+p^r}{2}$.

Remark 5.2.2. This algorithm was coded and compared with the one coded by applying Erhart's theory. The tables in Appendix 9 shows that our program runs faster than the other.

6. Reduced rings and geometry

6.1. **Reduced B_r-cohomology ring.** In Theorem 4.2.3, we computed $\mathrm{H}^\bullet(B_r, k)$ as a B-module. Describing its ring structure, however, is extremely hard for big r (see [AJ]). By looking at the reduced part, we can compute $\mathrm{H}^\bullet(B_r, k)$ as a finitely generated $\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}$-module. We first need the following observation.

Lemma 6.1.1. *Given any T-algebra M, there is an isomorphism*

$$(M^{T_r})_{\mathrm{red}} \cong (M_{\mathrm{red}})^{T_r}.$$

Proof. Note that $M_{\mathrm{red}} = M / \operatorname{Nilrad} M$, we consider the short exact sequence

$$0 \to \operatorname{Nilrad} M \to M \to M_{\mathrm{red}} \to 0.$$

As T_r is diagonalisable, $(-)^{T_r}$ is exact, so we obtain that

$$0 \to (\operatorname{Nilrad} M)^{T_r} \to (M)^{T_r} \to (M_{\mathrm{red}})^{T_r} \to 0.$$

On the other hand, we know that $(M^{T_r})_{\mathrm{red}} = M^{T_r} / \operatorname{Nilrad}(M^{T_r})$. So we just need to check that $(\operatorname{Nilrad} M)^{T_r} = \operatorname{Nilrad}(M^{T_r})$ which is true since both equal to $\operatorname{Nilrad}(M) \cap M^{T_r}$. ☐

From Corollary 4.1.2, we can identify $\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}$ with the polynomial algebra $k[x_1, \ldots, x_r]$ where each x_i is of weight $p^i \alpha$. The preceding lemma and Corollary 4.1.2 imply that

$$\begin{aligned} \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} &\cong \left(\mathrm{H}^\bullet(U_r, k)^{T_r}\right)_{\mathrm{red}} \cong \left(\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}\right)^{T_r} \\ &\cong \left(S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)}\right)^{T_r} \\ &\cong \left(k[x_1, \ldots, x_r]\right)^{T_r}. \end{aligned}$$

As a B-module, this reduced cohomology ring can be represented in terms of monomials as follows.

Theorem 6.1.2. *For $r \geq 1$, there is a $B^{(r)}$-module isomorphism*

$$\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \cong \bigoplus \langle x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r} \rangle \tag{8}$$

where the a_i are non-negative integers satisfying

$$a_1 + a_2 p + \cdots + a_r p^{r-1} = n p^{r-1} \tag{9}$$

for some $n \in \mathbb{N}$. Furthermore, let $R = k[x_1^{p^{r-1}}, x_2^{p^{r-2}}, \ldots, x_r]$, then the reduced ring $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ is a finitely free R-module with the basis $\mathfrak{B}_r = \{x_1^{a_1}, x_2^{a_2}, \ldots, x_{r-1}^{a_{r-1}}\}$ where for each $i = 1, \ldots, r-1$, $0 \leq a_i < p^{r-i}$ and satifies the equation (9).

Proof. The right-hand side is obtained by setting $b_1 = b_2 = \cdots = b_r = 0$ in the Theorem 4.2.1. Now observe that every tuple $(m_1p^{r-1}, m_2p^{r-2}, \ldots, m_r)$ with m_i a non-negative integer is a solution for (9). It follows that $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ contains R as a subring. Moreover, it can be verified that every monomial in the right-hand side of the isomorphism (8) is uniquely written as a product of an element in R and a monomial in $\mathfrak{B}_r$. The fact that $\mathfrak{B}_r$ is finite completes our proof. □

Corollary 6.1.3. *For each $r \geq 1$, there is a homeomorphism from* $\mathrm{Spec}\ k[x_1, \ldots, x_r]$ *onto* $\mathrm{Spec}\ \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$.

Proof. We recall the r-th Frobenius homomorphism of rings

$$\mathcal{F} : k[x_1, \ldots, x_r] \to k[x_1, \ldots, x_r]$$
$$\mathcal{F}(f) \longmapsto f^{p^{r-1}}$$

for all $f \in k[x_1, \ldots, x_r]$. Observe that

$$\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \subseteq k[x_1, \ldots, x_r]$$

are finitely generated commutative algebras. Note further that $\mathrm{Im}\,\mathcal{F} = k[x_1^{p^{r-1}}, \ldots, x_r^{p^{r-1}}]$ lies in the ring $R = k[x_1^{p^{r-1}}, x_2^{p^{r-2}}, \ldots, x_r]$; hence is a subalgebra of $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$. The inclusions

$$\mathrm{Im}\,\mathcal{F} \subseteq \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \subseteq k[x_1, \ldots, x_r]$$

implies that the morphism $\mathfrak{i} : \mathrm{Spec}\ k[x_1, \ldots, x_r] \to \mathrm{Spec}\ \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ is an F-isomorphism. Hence, it is a homeomorphism. □

Remark 6.1.4. One can easily construct an example where this morphism is not an isomorphism. From the computation of Andersen and Jantzen [AJ, 2.4], we have $\mathrm{Spec}\ \mathrm{H}^\bullet(B_2, k)_{\mathrm{red}} = \mathrm{Spec}\ k[x_1^p, x_2]$ which is obviously not isomorphic to $\mathrm{Spec}\ k[x_1, x_2]$ as there is no degree one morphism from one to the other.

6.2. **Reduced G_r-cohomology ring.** We first develop some techniques to compute reduced rings in general context.

Lemma 6.2.1. *Let k be a perfect field. Suppose G is a split reductive group over k and let M be a k-algebra. Then we have* $\mathrm{Nilrad}(k[G] \otimes M) = k[G] \otimes \mathrm{Nilrad}\, M$.

Proof. Consider the short exact sequence

$$0 \to \mathrm{Nilrad}\, M \to M \to M_{\mathrm{red}} \to 0.$$

As G is reductive, the coordinate algebra $k[G]$ is free over k [Jan, II.1.1]. So we have the following sequence

$$0 \to k[G] \otimes \mathrm{Nilrad}\, M \to k[G] \otimes M \to k[G] \otimes M_{\mathrm{red}} \to 0$$

is exact. It follows that

$$k[G] \otimes M_{\mathrm{red}} \cong \frac{k[G] \otimes M}{k[G] \otimes \mathrm{Nilrad}\, M}.$$

On the other hand, since $k[G]$ is reduced, it is well-known that the ring $k[G] \otimes M_{\mathrm{red}}$ is reduced when k is perfect. This implies that $k[G] \otimes \mathrm{Nilrad}\, M = \mathrm{Nilrad}\,(k[G] \otimes M)$. □

Lemma 6.2.2. *Suppose the same assumptions on k and G as in Lemma 6.2.1. Given a B-algebra M and suppose that all weights of* $\mathrm{Nilrad}\, M$ *are dominant. Then, as an algebra, we always have* $\left[\mathrm{ind}_B^G M\right]_{\mathrm{red}} \cong \mathrm{ind}_B^G(M_{\mathrm{red}})$.

Proof. We first show that $\left[(k[G]\otimes M)^B\right]_{\text{red}} \cong \left[(k[G]\otimes M)_{\text{red}}\right]^B$. Let $A = k[G]\otimes M$. Then we need to prove $(A^B)_{\text{red}} \cong (A_{\text{red}})^B$, i.e., $\frac{A^B}{\text{Nilrad}(A^B)} \cong \left(\frac{A}{\text{Nilrad}(A)}\right)^B$. This is equivalent to showing that the following sequence

$$0 \to \text{Nilrad}(A)^B \to A^B \to (A_{\text{red}})^B$$

is right exact; hence equivalent to $\mathrm{H}^1(B, \text{Nilrad}(A)) = 0$. Indeed, the preceding lemma shows that

$$\text{Nilrad}(A) = \text{Nilrad}(k[G]\otimes M) = k[G]\otimes \text{Nilrad}(M).$$

It follows by [Jan, Proposition I.4.10] and Kempf's vanishing that

$$\mathrm{H}^1(B, \text{Nilrad}(A)) = \mathrm{H}^1\left(B, k[G]\otimes \text{Nilrad}(M)\right) \cong R^1 \operatorname{ind}_B^G\left(\text{Nilrad}(M)\right) = 0$$

since all weights of Nilrad M are dominant. Finally, we have

$$\begin{aligned}\left(\operatorname{ind}_B^G M\right)_{\text{red}} &\cong \left[(k[G]\otimes M)^B\right]_{\text{red}} \\ &\cong \left[(k[G]\otimes M)_{\text{red}}\right]^B \\ &\cong \left[k[G]\otimes M_{\text{red}}\right]^B \\ &= \operatorname{ind}_B^G(M_{\text{red}}).\end{aligned}$$

∎

Now we are back to the assumption $G = SL_2$. The following result provides a link between the reduced parts of B_r- and of G_r-cohomology.

Theorem 6.2.3. *For each $r \geq 1$, there is a G-isomorphism*

$$\mathrm{H}^\bullet(G_r, k)_{\text{red}}^{(-r)} \cong \operatorname{ind}_B^G \mathrm{H}^\bullet(B_r, k)_{\text{red}}^{(-r)}.$$

Proof. It immediately follows from Theorem 4.3.1 and the lemma above. ∎

Proposition 6.2.4. *For each $r \geq 1$, there is a homeomorphism from* Spec $k[G\times^B \mathfrak{u}^r]$ *onto* Spec $\mathrm{H}^\bullet(G_r, k)_{\text{red}}$.

Proof. Let $\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)} = \text{Spec } k[x_1,\ldots,x_r]$ where each $\mathfrak{u}^{(i)}$ is identified with the weight space $k_{p^i\alpha}$; hence we consider it as an affine space equipped with the B-action. In Corollary 6.1.3, we can see that the F-isomorphism $\mathfrak{i}$ arises from the inclusion $\mathrm{H}^\bullet(B_r, k)_{\text{red}} \subseteq k[x_1,\ldots,x_r]$, so it is compatible with the B-action. It follows the B-equivariant homeomorphism

$$\mathfrak{i}^{(-r)} : \left(\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)}\right)^{(-r)} \to \text{Spec } \mathrm{H}^\bullet(B_r, k)_{\text{red}}^{(-r)}.$$

Apply the fibered product with G over B on both sides, we have a homeomorphism

$$id_G\times_B \mathfrak{i}^{(-r)} : G\times_B \left(\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)}\right)^{(-r)} \to G\times_B \text{Spec } \mathrm{H}^\bullet(B_r, k)_{\text{red}}^{(-r)}.$$

On the other hand, define a map:

$$\begin{aligned}\Phi : \mathfrak{u}^{(1-r)}\times\cdots\times\mathfrak{u} &\to \mathfrak{u}^r \\ (y_1, y_2, \ldots, y_r) &\mapsto (y_1^{p^{r-1}}, y_2^{p^{r-2}}, \ldots, y_r)\end{aligned}$$

for all $y_i \in \mathfrak{u}$. It is easy to see that Φ is a B-equivariant continuous map which is also a homeomorphism. It follows that the fibered products $G\times_B \left(\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)}\right)^{(-r)}$ and $G\times_B \mathfrak{u}^r$ are homeomorphic as a topological space. Now combine two above homeomorphisms and apply Theorem 6.2.3, we establish a homeomorphism from Spec $k[G\times^B \mathfrak{u}^r]$ onto Spec $\mathrm{H}^\bullet(G_r, k)_{\text{red}}$; hence completes our proof. ∎

7. Cohen-Macaulay Cohomology ring

It is known that there are many classes of groups for which the cohomology is Cohen-Macaulay [Ben2]. Our calculations in this section provide an evidence for the conjecture that simple algebraic groups also have Cohen-Macaulay cohomology. In particular, we show that the reduced rings $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}$ and $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$ are Cohen-Macaulay.

7.1. First we restate some facts in [JN].

Lemma 7.1.1. *Suppose $\lambda\in\mathbb{Z}\Phi$. Then for each $n\in\mathbb{Z}$ there is an G-isomorphism*

$$\mathrm{ind}_B^G(S^n(\mathfrak{u}^*)\otimes\lambda)\cong k[G\times^B\mathfrak{u}]_{(2n+2ht(\lambda))}.$$

Moreover, the algebra $\mathrm{ind}_B^G(S^n(\mathfrak{u}^*)\otimes\lambda)$ is $k[G\times^B\mathfrak{u}]$ shifted by the degree $2\mathrm{ht}(\lambda)$.

Proof. This is a consequence of Proposition 8.22 in [JN]. Note that since $\lambda\in\mathbb{Z}\Phi$, we have $\bar{\lambda}=0$. Then replace $k[\hat{\mathcal{O}},0]=k[\mathcal{O}]$ by $k[G\times^B\mathfrak{u}]$ in the formula in page 111. $\square$

Lemma 7.1.2. *For each $\lambda\in\mathbb{Z}\Phi$, the algebra $\mathrm{ind}_B^G\left(S^\bullet(\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)})\otimes\lambda\right)$ is isomorphic to $\mathrm{ind}_B^G S^\bullet(\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)})$ with the degree shifted by $2\mathrm{ht}(\lambda)$.*

Proof. We employ the algebra map in the previous lemma. In particular, for each degree n, we first consider

$$\begin{aligned}\mathrm{ind}_B^G\left(S^n(\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)})\otimes\lambda\right)&=\bigoplus_{a_1+\cdots+a_r=n}\mathrm{ind}_B^G\left(S^{a_1}(\mathfrak{u}^{*(1)})\otimes\cdots\otimes S^{a_r}(\mathfrak{u}^{*(r)})\otimes\lambda\right)\\&\cong\bigoplus_{a_1+\cdots+a_r=n}\mathrm{ind}_B^G\left((a_1p+\cdots+a_rp^r)\alpha+\lambda\right)\\&=\bigoplus_{a_1+\cdots+a_r=n}\mathrm{ind}_B^G\left(S^{(a_1p+\cdots+a_rp^r)}(\mathfrak{u}^*)\otimes\lambda\right)\\&\cong\bigoplus_{a_1+\cdots+a_r=n}k[G\times^B\mathfrak{u}]_{2(a_1p+\cdots+a_rp^r)+2ht(\lambda)}.\end{aligned}$$

Now if $\lambda=0$ in this observation, we obtain for each n that

$$\mathrm{ind}_B^G\left(S^n(\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)})\right)\cong\bigoplus_{a_1+\cdots+a_r=n}k[G\times^B\mathfrak{u}]_{2(a_1p+\cdots+a_rp^r)}.$$

This implies our proof. Consequently, the two algebras in the lemma are isomorphic (without the grading). $\square$

Lemma 7.1.3. *The ring $R=\mathrm{ind}_B^G\left(S^\bullet(\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)})\right)$ is Cohen-Macaulay.*

Proof. It is not hard to see that $k[\mathfrak{u}^r]$ is a free $k[\mathfrak{u}^{*(1)}\times\cdots\times\mathfrak{u}^{*(r)}]$-module. As the induction functor preserves direct sums, we have $k[G\times^B\mathfrak{u}^r]$ is a free R-module. Hence there is a flat homomorphism of R-modules

$$R\hookrightarrow k[G\times^B\mathfrak{u}^r]$$

By [Ngo, Theorem 5.2.7 - 5.4.1], we obtain that the ring $k[G\times^B\mathfrak{u}^r]$ is Cohen-Macaulay. As the flatness is locally preserved, Proposition 2.6(d) in [Ion] implies that the Cohen-Macaulayness of R follows from that of $k[G\times^B\mathfrak{u}^r]$. $\square$

We can now establish the goal of this section.

Theorem 7.1.4. *Both rings $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}$ and $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$ are Cohen-Macaulay.*

Proof. Note that $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}} = k[\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)}]^{T_r}$ an invariant of a regular domain by a finite group scheme T_r. So it is Cohen-Macaulay by a famous result of Hochster and Roberts [HR].

Next by Theorem 6.2.3, we have the decomposition of R-modules

$$\begin{aligned}\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}} &\cong \operatorname{ind}_B^G \mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}\\ &\cong \bigoplus_{x_\lambda\in\mathfrak{B}_r} \operatorname{ind}_B^G\left(x_\lambda\otimes k[x_1,\dots,x_r]\right).\end{aligned}$$

In other words, the ring $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$ is a free R-module. By the previous lemma, R is Cohen-Macaulay, so is $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$. □

7.2. **Open questions.** The results above imply many open questions involving the properties in commutative algebra like Cohen-Macaulayness for the objects in representation theory as follows.

Conjecture 7.2.1. *Suppose R is a B-algebra. If R is Cohen-Macaulay, then so is $\operatorname{ind}_B^G R$.*

Conjecture 7.2.2. *Let G be a simple algebraic group defined over an algebraically closed field k of characteristic p, a good prime for G. Then both $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}$ and $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$ are Cohen-Macaulay.*

Evidently, if $r=1$ then the results of Andersen and Jantzen [AJ], Friedlander and Parshall [FP] show that $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}\cong S^\bullet(\mathfrak{u}^*)$ is regular and $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}\cong k[\mathcal{N}]$ is Cohen-Macaulay. Although our computation supports the conjecture in the case $G=SL_2$ for arbitrary r, it is a difficult problem as very little appears to be known about cohomology of higher Frobenius kernels.

8. Quantum calculations

In this section, we apply our methods to compute cohomology for the Frobenius–Lusztig kernels of quantum groups defined in [Dru]. We first recall the definitions as follows. (Details can be found in [Dru, 2.1-2.2].)

8.1. **Notation and Construction.** We basically recover the material in [Dru, Section 2.1] in the case $\mathfrak{g}=\mathfrak{sl}_2$. Let k be a field of characteristic $p\neq 2$. Let ℓ be an odd positive integer not divisible by p. Denote by $\Phi=\{\alpha,-\alpha\}$ the root system of $\mathfrak{sl}_2$.

Let q be an indeterminate and let $\mathbb{U}_q$ be the quantized enveloping algebra associated to $\mathfrak{sl}_2$, which is the $\mathbb{Q}(q)$-algebra defined by the generators E,F,K,K^{-1} and satisfying the relations

$$\begin{aligned}KK^{-1} &= 1 = K^{-1}K,\\ KEK^{-1} &= q^2E,\\ KFK^{-1} &= q^{-2}F,\\ EF-FE &= \frac{K-K^{-1}}{q-q^{-1}}.\end{aligned}$$

Set $\mathsf{A}=\mathbb{Z}[q,q^{-1}]$. For given integer i, set

$$[i]=\frac{q^i-q^{-i}}{q-q^{-1}},$$

then denote $[i]^! = [i][i-1]\dots[1]$. Note that $[0]^!=1$ as a convention. Suppose m is a positive integer and n is an integer, we write

$$\begin{bmatrix} n\\ m\end{bmatrix} = \frac{[n][n-1]\dots[n-m+1]}{[1][2]\dots[m]}$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$. Next, for each $m \geq 0$, we define the m-th divided powers as follows:

$$E^{(m)} = \frac{E^m}{[m]^!}, \quad F^{(m)} = \frac{F^m}{[m]^!}.$$

Now the Lusztig A-form quantized enveloping algebra U_{A} is defined as the A-subalgebra of $\mathbb{U}_q$ generated by $\{E^{(n)}, F^{(n)}, K^{\pm 1} : n \in \mathbb{N}\}$. Fix $\zeta \in k$ a primitive ℓ-th root of unity in k. We consider k as A-algebra by the homomorphism $\mathbb{Z}[q, q^{-1}] \to k$ mapping $q \mapsto \zeta$. Let

$$U_\zeta = \frac{U_k}{\langle K^\ell \otimes 1 - 1 \otimes 1 \rangle}$$

where $U_k = U_{\mathsf{A}} \otimes_{\mathsf{A}} k$. Denote by u_k the Hopf subalgebra of U_k generated by $\{E, F, K\}$. Let u_ζ be the image of u_k in U_ζ, and call it the small quantum group.

For each $r \geq 1$, we define $U_\zeta(G_r)$ to be the subalgebra of U_ζ generated by

$$\{E, E^{(p^i\ell)}, F, F^{(p^i\ell)}, K : 0 \leq i \leq r-1\},$$

and call it the r-th Frobenius-Lusztig kernel of U_ζ. Note that if $p = 0$ then we obtain for every $r \geq 1$ that $U_\zeta(G_r) = u_\zeta$. We also define for each $r \geq 1$

$$U_\zeta(B_r) = U_\zeta(B) \cap U_\zeta(G_r) = \left\langle E, E^{(p^i\ell)}, K \ : \ 0 \leq i \leq r-1 \right\rangle,$$
$$U_\zeta(U_r) = U_\zeta(U) \cap U_\zeta(G_r) = \left\langle E, E^{(p^i\ell)} \ : \ 0 \leq i \leq r-1 \right\rangle.$$

Let $\mathrm{Dist}(G)$ be the algebra of distributions on G. It is known that there is an isomorphism of Hopf algebras between $U_\zeta//u_\zeta$ and $\mathrm{Dist}(G)$. The quotient map $F_\zeta : U_\zeta \to \mathrm{Dist}(G)$ is the quantum analog of the Frobenius homomorphism. Note that the restriction $F_\zeta : U_\zeta(T_r) \to \mathrm{Dist}(T_r)$ for each $r \geq 1$ induces an isomorphism $U_\zeta(T_r)//u_\zeta^0 \cong \mathrm{Dist}(T_r)$.

8.2. **Cohomology of $U_\zeta(U_r)$.** It is observed that for each $r \geq 1$, $U_\zeta(U_r)$ is a normal subalgebra of $U_\zeta(B_r)$. There is a right adjoint action of U_ζ^0 on $U_\zeta(B_r)$ such that $U_\zeta(U_r)$ is a U_ζ^0-submodule of $U_\zeta(B_r)$. Then from [Dru, Theorem 4.3.1], the cohomology $\mathrm{H}^\bullet(U_\zeta(U_r), k)$ is a left U_ζ^0-module. Before computing this module stucture, we need to revise some notation in Section 2. Let $S^\bullet(x_0, \ldots, x_r)$ be the symmetric algebra over $x_0, \ldots, x_r$ of degree 2. Then this symmetric algebra can be considered as a U_ζ^0-module by assigning weight $p^i\ell\alpha$ to x_i for each $0 \leq i \leq r$. Let $\Lambda^\bullet(y_0, \ldots, y_r)$ be the exterior algebra generated by $y_0, \ldots, y_r$ of degree 1. By assigning weight α to y_0 and weight $p^{i-1}\ell\alpha$ to y_i for all $1 \leq i \leq r$, we obtain the U_ζ^0-module structure of $\Lambda^\bullet(y_0, \ldots, y_r)$. Now we compute the cohomology of $U_\zeta(U_r)$ as follows.

Theorem 8.2.1. *For each $r \geq 1$, there is an isomorphism of U_ζ^0-algebras*

$$\mathrm{H}^\bullet(U_\zeta(U_r), k) \cong S^\bullet(x_0, \ldots, x_r) \otimes \Lambda^\bullet(y_0, \ldots, y_r).$$

Proof. Note that $U_\zeta(U_r)$ is in this case the same as $\mathrm{gr}U_\zeta(U_r)$, the associated graded algebra of $U_\zeta(U_r)$. According to [Dru], it is a k-algebra generated by E_α and $E_\alpha^{(p^i\ell)}$ for all $1 \leq i \leq r-1$, subject to the relations (6.1.2), (6.1.3), and (6.1.4) in [Dru] applied with $\Phi^+ = \{\alpha\}$. Hence, by Theorem 4.1 and Remark 4.2 in [MPSW], we obtain the isomorphism as desired. $\square$

Remark 8.2.2. This result is a special case of [Dru, Proposition 6.2.2] in which Drupieski computed the cohomology of $\mathrm{gr}U_\zeta(U_r)$ in a more general context.

8.3. **Cohomology of $U_\zeta(B_r)$.** Since $U_\zeta(B_r)//U_\zeta(U_r) \cong U_\zeta(T_r)$, for each $r \geq 1$, there exists a Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(U_\zeta(T_r), \mathrm{H}^j(U_\zeta(U_r), k)) \Rightarrow \mathrm{H}^{i+j}(U_\zeta(B_r), k).$$

As $U_\zeta(T_r)$ is a semisimple Hopf algebra, the spectral sequence collapses at the second page and then we obtain the following algebra isomorphism

$$\mathrm{H}^\bullet(U_\zeta(B_r), k)) \cong \mathrm{H}^0(U_\zeta(T_r), \mathrm{H}^\bullet(U_\zeta(U_r), k)) = \mathrm{H}^\bullet(U_\zeta(U_r), k)^{U_\zeta(T_r)}.$$

Note that the U_ζ^0-module stucture is preserved via this isomorphism. From Theorem 8.2.1, first taking the u_ζ^0-invariant of $\mathrm{H}^\bullet(U_\zeta(U_r), k)$, we have

$$\begin{aligned}\mathrm{H}^\bullet(U_\zeta(U_r), k)^{u_\zeta^0} &\cong (\Lambda^\bullet(y_0))^{u_\zeta^0} \otimes S^\bullet(x_0, \dots, x_r) \otimes \Lambda^\bullet(y_1, \dots, y_r) \\ &= S^\bullet(x_0, \dots, x_r) \otimes \Lambda^\bullet(y_1, \dots, y_r)\end{aligned}$$

since $\Lambda^\bullet(y_0)^{u_\zeta^0} = \Lambda^\bullet(y_0^\ell) = k$. Note also that each generator's weight in $\Lambda^\bullet$ is divided by ℓ under the Frobenius homomorphism. These results can be enclosed in computing the $U_\zeta(T_r)$-invariant of $U_\zeta(U_r)$-cohomology as follows.

$$\begin{aligned}\mathrm{H}^\bullet(U_\zeta(U_r), k)^{U_\zeta(T_r)} &\cong \left(\mathrm{H}^\bullet(U_\zeta(U_r), k)^{u_\zeta^0}\right)^{U_\zeta(T_r)//u_\zeta^0} \\ &\cong \left(S^\bullet(x_0, \dots, x_r) \otimes \Lambda_\zeta^\bullet(y_1, \dots, y_r)\right)^{U_\zeta(T_r)//u_\zeta^0} \\ &\cong \left(S^\bullet(x_0', x_1', \dots, x_r') \otimes \Lambda^\bullet(y_1', \dots, y_r')\right)^{\mathrm{Dist}(T_r)}\end{aligned}$$

where x_0' is of weight α, and each x_i' (or y_{i+1}') is of weight $p^i\alpha$ for all $0 \leq i \leq r$. Now we can use an analogous argument as in Subsection 4.2 to compute the $U_\zeta(B_r)$-cohomology.

Theorem 8.3.1. *For each $r \geq 1$, there is an isomorphism of T-modules*

$$\mathrm{H}^\bullet(U_\zeta(B_r), k)^{(-r)} \cong \bigoplus_{n \in \mathbb{N}} (n\alpha)^{N_r'(p,n)}$$

where $N_r'(p,n)$ is the number of solutions $(a_0, \dots, a_r, b_1, \dots, b_r) \in \mathbb{N}^{r+1} \times \{0,1\}^r$ for the equation

$$(a_0 + b_1) + p(a_1 + b_2) + \cdots + p^{r-1}(a_{r-1} + b_r) + p^r a_r = \frac{n}{2}p^r.$$

Remark 8.3.2. Comparing with the equation (5) in Section 4, we have

$$N_r'(p,n) = \sum_{m=0}^{\infty} N_r(p,m,n).$$

It is also observed that $N_r(p,m,n) = 0$ when $m > np^r$ so the sum above is well-defined and $N_r'(p,n)$ is finite.

8.4. **Cohomology of $U_\zeta(G_r)$.** We first need to develop a spectral sequence which is similar to the one in Subsection 3.2.

Lemma 8.4.1. *Let M be a $U_\zeta(B)$-module such that $R^i \operatorname{ind}_{U_\zeta(B)}^{U_\zeta} M = 0$ for each $i \geq 1$. Then there is for each $r \geq 1$ a spectral sequence of G-modules*

$$R^i \operatorname{ind}_B^G \mathrm{H}^j(U_\zeta(B_r), M)^{(-r)} \Rightarrow \left(\mathrm{H}^{i+j}(U_\zeta(G_r), \operatorname{ind}_{U_\zeta(B)}^{U_\zeta} M)\right)^{(-r)}.$$

Proof. Note that the category of rational B-modules (resp. G-modules) is equivalent to the category of integrable $\mathrm{Dist}(B)$-modules (resp. locally finite $\mathrm{Dist}(G)$-modules) [CPS, 6.9, 9.4]. Identifying $\mathrm{Dist}(H)$ with $U_\zeta(H)//u_\zeta$ via the restriction of Frobenius homomorphism F_ζ where H is G, B, G_r, or B_r. Let

$$\mathcal{F}_1(-) = \mathrm{ind}_B^G(-) \circ (-)^{U_\zeta(B_r)(-r)},$$
$$\mathcal{F}_2(-) = (-)^{U_\zeta(G_r)(-r)} \circ \mathrm{ind}_{U_\zeta(B)}^{U_\zeta}(-)$$

being functors from the category of $U_\zeta(B)$-modules to the category of rational G-modules. From [Jan, Proposition I.4.1], we have the following spectral sequences

$$R^i \mathrm{ind}_B^G \mathrm{H}^j(U_\zeta(B_r), M)^{(-r)} \Rightarrow R^{i+j}\mathcal{F}_1(M), \tag{10}$$

$$\mathrm{H}^i(U_\zeta(G_r), R^j \mathrm{ind}_{U_\zeta(B)}^{U_\zeta} M)^{(-r)} \Rightarrow R^{i+j}\mathcal{F}_2(M). \tag{11}$$

On the other hand, we consider

$$\mathrm{ind}_B^G(M^{U_\zeta(B_r)})^{(-r)} \cong \mathrm{ind}_B^G(M^{u_\zeta(B)})^{U_\zeta(B_r)//u_\zeta(B)(-r)} \cong \mathrm{ind}_B^G(M^{u_\zeta(B)})^{B_r(-r)}$$

which is isomorphic to the functor $\left(\mathrm{ind}_B^G M^{u_\zeta(B)}\right)^{G_r(-r)}$ by [AJ, 3.1]. In addition, we have

$$\begin{aligned}\left(\mathrm{ind}_B^G M^{u_\zeta(B)}\right)^{G_r} &\cong \left(\mathrm{ind}_B^G M^{u_\zeta(B)}\right)^{U_\zeta(G_r)//u_\zeta}\\ &\cong \left[\left(\mathrm{ind}_{U_\zeta(B)}^{U_\zeta} M\right)^{u_\zeta}\right]^{U_\zeta(G_r)//u_\zeta}\\ &\cong \left(\mathrm{ind}_{U_\zeta(B)}^{U_\zeta} M\right)^{U_\zeta(G_r)}\end{aligned}$$

where the second isomorphism is because of Theorem 5.1.1 in [Dru]. In summary, the two functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are naturally isomorphic. Hence, the spectral sequences (10), (11) converge to the same abutment. Since $R^i \mathrm{ind}_{U_\zeta(B)}^{U_\zeta} M = 0$ for each $i \geq 1$, the second spectral sequence collapses and then gives us the desired spectral sequence. □

Remark 8.4.2. As the isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$ respects the cup-products, similar argument as in [AJ, Remark 3.2] shows that the spectral sequence in the lemma above is compatible with the $\mathrm{H}^\bullet(U_\zeta(G_r), k)$-module structure.

Now we can compute the cohomology of $U_\zeta(G_r)$.

Theorem 8.4.3. *There are for each $r \geq 1$ G-module isomorphisms*

$$\mathrm{H}^\bullet(U_\zeta(G_r), k)^{(-r)} \cong \mathrm{ind}_B^G \mathrm{H}^\bullet(U_\zeta(B_r), k)^{(-r)} \cong \bigoplus_{n\in\mathbb{N}} \mathrm{ind}_B^G(n\alpha)^{N'_r(p,n)}$$

in which the first isomorphism is of graded k-algebras.

Proof. Note first that $R^i \mathrm{ind}_{U_\zeta(B)}^{U_\zeta} k = 0$ for all $i \geq 1$ by the quantum version of Kempf's Vanishing Theorem [RH, Theorem 5.5]. The preceding lemma implies the following spectral sequence

$$R^i \mathrm{ind}_B^G \mathrm{H}^j(U_\zeta(B_r), k)^{(-r)} \Rightarrow \mathrm{H}^{i+j}(U_\zeta(G_r), \mathrm{ind}_{U_\zeta(B)}^{U_\zeta} k)^{(-r)}.$$

On the other hand, Theorem 8.3.1 shows that $\mathrm{H}^j(U_\zeta(B_r), k)^{(-r)}$ is decomposed into dominant weight spaces for each $j \geq 0$. Hence, we have

$$R^i \mathrm{ind}_B^G \mathrm{H}^j(U_\zeta(B_r), k)^{(-r)} = 0$$

for all $i > 0$, which implies the collapse of the spectral sequence so that we have for each $j \geq 0$

$$\mathrm{ind}_B^G \mathrm{H}^j(U_\zeta(B_r), k)^{(-r)} \cong \mathrm{H}^j(U_\zeta(G_r), k)^{(-r)}$$

as a G-module. By Remark 8.4.2, these isomorphisms extend to the first isomorphism of the theorem. The second one immediately follows from Theorem 8.3.1. □

8.5. **Cohen-Macaulay quantum cohomology.** We continue establishing in this section analogous results as in Sections 6 and 7. As most of the arguments here are similar to those of the aforementioned sections, we omit details of proofs below.

We begin by looking at the reduced parts of quantum cohomology rings.

Proposition 8.5.1. *For each $r \geq 0$, there are isomorphisms of rings*

$$\mathrm{H}^\bullet(U_\zeta(B_r), k)_{\mathrm{red}} \cong S^\bullet(x_0, \ldots, x_r)^{T_r},$$
$$\mathrm{H}^\bullet(U_\zeta(G_r), k)_{\mathrm{red}} \cong \mathrm{ind}_B^G S^\bullet(x_0, \ldots, x_r)^{T_r}.$$

Proof. We have

$$\begin{aligned}
\mathrm{H}^\bullet(U_\zeta(B_r), k)_{\mathrm{red}} &\cong \left(\mathrm{H}^\bullet(U_\zeta(U_r), k)^{U_\zeta(T_r)}\right)_{\mathrm{red}} \\
&\cong \left((\mathrm{H}^\bullet(U_\zeta(U_r), k)^{u_\zeta^0})^{U_\zeta(T_r)//u_\zeta^0}\right)_{\mathrm{red}} \\
&\cong \left([S^\bullet(x_0, \ldots, x_r) \otimes \Lambda^\bullet(y_1, \ldots, y_r)]^{T_r}\right)_{\mathrm{red}} \\
&\cong \left([S^\bullet(x_0, \ldots, x_r) \otimes \Lambda^\bullet(y_1, \ldots, y_r)]_{\mathrm{red}}\right)^{T_r} \\
&\cong S^\bullet(x_0, \ldots, x_r)^{T_r}
\end{aligned}$$

where the fourth isomorphism is from Lemma 6.1.1. Next, Lemma 6.2.2 gives us the second isomorphism of the proposition. □

Now the following theorem shows that the Frobenius-Lusztig kernels of quantum groups also have Cohen-Macaulay cohomology.

Theorem 8.5.2. *For each $r \geq 0$, the cohomology rings $\mathrm{H}^\bullet(U_\zeta(B_r), k)_{\mathrm{red}}$ and $\mathrm{H}^\bullet(U_\zeta(G_r), k)_{\mathrm{red}}$ are Cohen-Macaulay.*

Proof. The argument is similar to that given for Theorem 7.1.4. □

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9. Appendix

9.1. In this section we compare the effectiveness of two programs computing the number $N_r(p, m, n)$. The first program is encoded from the algorithm in Section 5 and denote by F_1 in the tables below. The other one is applied Ehrhart's theory on counting integral lattice points in a polytope and denote by F_2.[1] Note also that both programs are written in MAGMA code and run on the MAGMA computer algebra system.

In both tables, we fix $m = 0$ and respectively consider $p = 3$ in Table 1, and $p = 5$ in Table 2. The last two columns show the time needed to compute $N_r(p, m, 0)$ for all $0 \leq m \leq 10$. The results show that $N_r(p, 0, n) = 0$ if n is odd, hence only results for m even are exhibited in the tables.

Note that the symbol '...' indicates that the program is still running.

[1]Both codes are available on arxiv.org.

TABLE 1. $p = 3$

$r\backslash n$	0	2	4	6	8	10	F_1	F_2
2	1	3	5	7	9	11	0	1.56 (s)
3	1	13	37	73	121	181	0	5.74 (s)
4	1	111	545	1519	3249	5951	0.01 (s)	41.04 (s)
5	1	2065	17857	70705	195601	439201	0.77 (s)	67392.07 (s)
6	1	88563	1387589	7853335	28165977	77619179	97.21 (s)	... (s)
7	1	9240925	266869765	2169510553	10107924457	34221999925	38755.21 (s)	... (s)

TABLE 2. $p = 5$

$r\backslash n$	0	2	4	6	8	10	F_1	F_2
2	1	3	5	7	9	11	0	1.86 (s)
3	1	21	61	121	201	301	0	7.15 (s)
4	1	503	2505	7007	15009	27511	0.08 (s)	43.86 (s)
5	1	42521	377561	1505121	4175201	9387801	16.78 (s)	69773.82 (s)
6	1	13563003	224065005	1283632007	4625640009	12777215011	16511.43 (s)	... (s)
7	1						... (s)	... (s)

REFERENCES

[AJ] H. H. Andersen and J. C. Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann. **269** (1984), 487–525.

[Bar] A. I. Barvinok, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math. Operations Research **19** (1994), 769–779.

[BNP1] C. P. Bendel, D. K. Nakano, and C. Pillen, *Extensions for frobenius kernels*, J. Algebra **272** (2004), 476–511.

[BNP2] ———, *Second cohomology groups for frobenius kernels and related structures*, Adv. Math. **209** (2006), 162–197.

[Ben1] D. J. Benson, *Representations and cohomology. II*, second ed., Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, Cambridge, 1998, Cohomology of groups and modules.

[Ben2] D. J. Benson, *Commutative algebra in the cohomology of groups*, Trends in Com. Algebra **51** (2004).

[Car] J. F. Carlson, *Systems of parameters and the structure of cohomology rings of finite groups*, Contemp. Math. **158** (1994), 1–7.

[CPS] E. Cline, B. Parshall, and L. Scott, J. Algebra **63**.

[Dru] C. M. Drupieski, *Representations and cohomology for frobenius-lusztig kernels*, J. Pure Appl. Algebra **215** (2011), 1473–1491.

[FP] E. M. Friedlander and B. J. Parshall, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. **108** (1986), 235–253.

[HR] M. Hochster and J. L. Roberts, *Actions of reductive groups on regular rings and cohen-macaulay rings*, Bull. AMS. **80** (1974), 281–284.

[Hum] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised.

[Ion] C. Ionescu, *Cohen-macaulay fibres of a morphism*, AAPP **86** (2008), 1–9.

[JN] J. C. Jantzen and K.-H. Neeb, *Lie theory: Lie algebras and representations*, Progress in Mathematics, vol. 228, Birkhauser, 2004.

[Jan] J. C. Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[MPSW] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon, *Cohomology of finite dimensional pointed hopf algebras*, Proc. London Math. Soc. **100** (2010), 377–404.

[Ngo] N. V. Ngo, *Commuting varieties of r-tuples over Lie algebras*, preprint, 2012.
[RH] S. Ryom-Hansen, *A q-analogue of kempf s vanishing theorem*, Mosc. Math. J. **3** (2003), 173–187.
[SFB1] A. Suslin, E. M. Friedlander, and C. P. Bendel, *Infinitesimal 1-parameter subgroups and cohomology*, J. Amer. Math. Soc **10** (1997), 693–728.
[SFB2] ———, *Support varieties for infinitesimal group scheme*, J. Amer. Math. Soc **10** (1997), 729–759.
[vdK] W. van der Kallen, *Infinitesimal fixed points in modules with good filtration*, Math. Z. **212** (1993), 157–159.

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